

From Clifton-Barrow spherically symmetric to axially symmetric solution in $f(R)$ -gravity

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Starting from Clifton's spherical static solution for $f(R)$ -gravity we can derive axial symmetric solutions using a methods that takes the advantage of a complex coordinate transformation. The new solutions are generated by applying the Newman-Janis algorithm previously used to transform any static spherically symmetric into axially metric in General Relativity .

I. INTRODUCTION

Newman and Janis showed that it is possible to obtain an axially symmetric solution (like the Kerr metric) by making an elementary complex transformation on the Schwarzschild solution [1]. This same method has been used to obtain a new stationary and axially symmetric solution known as the Kerr-Newman metric [2]. The Kerr-Newman space-time is associated to the exterior geometry of a rotating massive and charged black-hole. For a review on the Newman-Janis method to obtain both the Kerr and Kerr-Newman metrics see [3].

By means of very elegant mathematical arguments, Schiffer et al. [4] have given a rigorous proof to show how the Kerr metric can be derived starting from a complex transformation on the Schwarzschild solution. We will not go into the details of this demonstration, but point out that the proof relies on two main assumptions. The first is that the metric belongs to the same algebraic class of the Kerr-Newman solution, namely the Kerr-Schild class [5]. The second assumption is that the metric corresponds to an empty solution of the Einstein field equations. In the case we are going to study, these assumptions are not considered and hence the proof in [4] is not applicable. It is clear, by the generation of the Kerr-Newman metric, that all the components of the stress-energy tensor need to be non-zero for the Newman-Janis method to be successful. In fact, Gürses and Gürsey, in 1975 [6], showed that if a metric can be written in the Kerr-Schild form, then a complex transformation "is allowed in General Relativity." In this paper, we will show that such a transformation can be extended to $f(R)$ -gravity as has already been done by the author in a previous paper [7]. The interest in spherically and axial symmetric solutions of $f(R)$ -gravity is growing up because, they could lead to solutions to determine the black holes [8, 9]. Exact solutions of the field equations are important in order to gain insight into the mathematical and physical content of a theory [10], and can be obtained only upon assuming special symmetries. In [11], solutions in vacuum have been found considering relations among functions that define the spherical metric or imposing a constant Ricci curvature scalar. The authors have reconstructed the form of some $f(R)$ -models,

discussing their physical relevance. In [12], the same authors have discussed static spherically symmetric solutions, in presence of perfect fluid matter, adopting the metric formalism. They have shown that a given matter distribution is not capable of globally determining the functional form of $f(R)$. Others authors have discussed in details the spherical symmetry of $f(R)$ -gravity considering also the relations with the weak field limit. Exact solutions are obtained for constant Ricci curvature scalar and for Ricci scalar depending on the radial coordinate. In particular, it can be considered how to obtain results consistent with GR assuming the well-known post-Newtonian and post-Minkowskian limits as consistency checks. [14]. In the case of metric $f(R)$ theory *in vacuo*, assuming spherical symmetry does not automatically lead to the Schwarzschild or Schwarzschild-de Sitter solutions [14]. The class of spherically symmetric solutions of metric $f(R)$ gravity is still unexplored, with only a handful of analytical solutions beyond Schwarzschild-de Sitter being known [8, 9]. When one considers axially symmetric solutions, the situation is even worse: apart from the Kerr metric [15], only the solutions of Ref. [7] are presently available.

We consider, in particular, a static spherical solution discovered by Clifton and Barrow in $f(R) = R^{1+\delta}$ gravity [13], which does not resemble the Schwarzschild-de Sitter solution, and use it as a seed metric to generate a new rotating solution which is not the Kerr metric or a known generalization of it. The outline of this paper is as follows. In the Sec.II, we introduce the $f(R)$ -gravity action and the field equations with its spherical solutions. In Sec.III, the Newman-Janis method is applied to Clifton-Barrow solution. Discussion and concluding remarks are drawn in Sec. IV.

II. FIELD EQUATIONS AND CLIFTON-BARROW SPHERICALLY SYMMETRIC SOLUTIONS

Let us consider a function $f(R)$ of the Ricci scalar R in four dimensions [16]. The gravitational action is

$$\mathcal{A} = \int d^4x \sqrt{-g} \left[f(R) + \mathcal{X} \mathcal{L}_m \right], \quad (1)$$

where $\mathcal{X} = \frac{16\pi G}{c^4}$ is the coupling, \mathcal{L}_m is the standard matter Lagrangian and g is the determinant of the metric¹. The field equations, in metric formalism, read²

$$f'(R)R_{\mu\nu} - \frac{1}{2}f g_{\mu\nu} - f'(R)_{;\mu\nu} + g_{\mu\nu}\square_g f'(R) = \frac{\mathcal{X}}{2}T_{\mu\nu}, \quad (2)$$

$$3\square f'(R) + f'(R)R - 2f(R) = \frac{\mathcal{X}}{2}T, \quad (3)$$

with $T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}$ the energy momentum tensor of matter (T is the trace), $f'(R) = \frac{df(R)}{dR}$ and $\square_g = ;_{\sigma}{}^{\sigma}$. We adopt a $(+, -, -, -)$ signature, while the conventions for Ricci's tensor is $R_{\mu\nu} = R^{\sigma}{}_{\mu\sigma\nu}$ and $R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}_{\beta\nu,\mu} + \dots$ for the Riemann tensor, where

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}g^{\mu\sigma}(g_{\alpha\sigma,\beta} + g_{\beta\sigma,\alpha} - g_{\alpha\beta,\sigma}), \quad (4)$$

are the Christoffel symbols of the $g_{\mu\nu}$ metric. We choice $f(R) = R^{1+\delta}$ which reduces to General Relativity in the limit $\delta \rightarrow 0$, and we consider only the exact static spherically symmetric vacuum solutions deriving from this Lagrangian [13]. The static solution is given by the line elements

$$ds^2 = A(r)dt^2 - \frac{dr^2}{B(r)} - r^2 d\Omega^2, \quad (5)$$

where

$$A(r) = r^{2\delta \frac{1+\delta}{1-\delta}} + \frac{C}{r^{\frac{1-4\delta}{1-\delta}}},$$

$$B(r) = \frac{(1-\delta)^2}{(1-2\delta+4\delta^2)(1-2\delta(1+\delta))} \left(1 + \frac{C}{r^{\frac{1-2\delta+4\delta^2}{1-\delta}}} \right),$$

and C is a constant. The task is now to show how, from a spherically symmetric solution, one can generate an axially symmetric solution adopting the Newman-Janis procedure that works in General Relativity.

III. AXIAL SYMMETRY FOR $f(R) = R^{1+\delta}$ GRAVITY

We want to show, now, how it is possible to obtain an axially symmetric solution starting from a spherically symmetric seed solution (5) adopting the method developed by Newman and Janis in General Relativity [1, 7]. Such an algorithm can be applied to a static spherically symmetric metric, considered as a “seed” metric. Following Newman and Janis, Eq. (5) can be written in the so called Eddington–Finkelstein coordinates (u, r, θ, φ) , *i.e.* the g_{rr} component is eliminated by a change of coordinates and a cross term is introduced [17]. Specifically this is achieved by defining the time coordinate as $dt = du + F(r)dr$ and setting

$$F(r) = \left[\frac{A(r)}{B(r)} \right]^{-\frac{1}{2}} = \frac{r^{2\delta-1}}{\sqrt{\Psi} \left(C + r^{\frac{1-2\delta+4\delta^2}{1-\delta}} \right)}, \quad (6)$$

where for simplicity we set $\Psi = \frac{(1-\delta)^2}{(1-2\delta+4\delta^2)[1-2\delta(1+\delta)]}$. Once such a transformation is performed, the metric (5) becomes

$$ds^2 = A(r) du^2 + 2\sqrt{A(r)B(r)} du dr - r^2 d\Omega^2 =$$

$$= A(r) du^2 + 2r^{\frac{\delta(2\delta+1)}{1-\delta}} (\Psi)^{-\frac{1}{2}} du dr - r^2 d\Omega^2. \quad (7)$$

The surface $u = \text{costant}$ is a light cone starting from the origin $r = 0$. The metric tensor for the line element (7) in contravariant form and null-coordinates is

$$ds^2 = 2(\Psi)^{-\frac{1}{2}} r^{\frac{\delta(2\delta+1)}{1-\delta}} du dr -$$

$$+ \Psi \left[1 + C \left(r^{\frac{1-2\delta+4\delta^2}{1-\delta}} \right)^{-1} \right] dr^2 - \frac{1}{r^2} d\theta^2 -$$

$$+ \frac{1}{r^2 \sin^2 \theta} d\varphi^2 \quad (8)$$

The metric (8) can be written in terms of a null tetrad as

$$g^{\mu\nu} = l^{\mu}n^{\nu} + l^{\nu}n^{\mu} - m^{\mu}\bar{m}^{\nu} - m^{\nu}\bar{m}^{\mu}, \quad (9)$$

where l^{μ} , n^{μ} , m^{μ} and \bar{m}^{μ} are the vectors satisfying the conditions

$$l_{\mu}l^{\mu} = m_{\mu}m^{\mu} = n_{\mu}n^{\mu} = 0, \quad (10)$$

$$l_{\mu}n^{\mu} = -m_{\mu}\bar{m}^{\mu} = 1, \quad (11)$$

$$l_{\mu}m^{\mu} = n_{\mu}\bar{m}^{\mu} = 0. \quad (12)$$

¹ Here we indicates with “,” partial derivative and with “;” covariant derivative with regard to $g_{\mu\nu}$; all Greek indices run from 0, ..., 3 and Latin indices run from 1, ..., 3; g is the determinant.

² All considerations are developed here in metric formalism. From now on we assume physical units $G = c = 1$.

The bar indicates the complex conjugation. At any point in space, the tetrad can be chosen in the following manner: l^μ is the outward null vector tangent to the cone, n^μ is the inward null vector pointing toward the origin, and m^μ and \bar{m}^μ are the vectors tangent to the two-dimensional sphere defined by constant r and u (see [18]). For the space-time (8), the tetrad null vectors can be

$$\begin{cases} l^\mu = \delta_1^\mu, \\ n^\mu = -\frac{1}{2} \left[\Psi \left(1 + \frac{C}{r \frac{1-2\delta+4\delta^2}{\delta-1}} \right) \right] \delta_1^\mu + \frac{\sqrt{\Psi}}{r \frac{\delta(2\delta+1)}{1-\delta}} \delta_0^\mu, \\ m^\mu = \frac{1}{\sqrt{2}r} \left(\delta_2^\mu + \frac{i}{\sin \theta} \delta_3^\mu \right), \\ \bar{m}^\mu = \frac{1}{\sqrt{2}r} \left(\delta_2^\mu - \frac{i}{\sin \theta} \delta_3^\mu \right). \end{cases} \quad (13)$$

Now we need to extend the set of coordinates $x^\mu = (u, r, \theta, \varphi)$ replacing the real radial coordinate by a complex variable. Then the tetrad null vectors become³

$$\begin{cases} l^\mu = \delta_1^\mu, \\ n^\mu = -\frac{1}{2} \Psi \left[\frac{(r\bar{r})^\beta + C(r+\bar{r})^\beta}{(r\bar{r})^\beta} \right] \delta_1^\mu + \frac{\sqrt{\Psi}}{(r\bar{r})^{\frac{\delta(2\delta+1)}{2(1-\delta)}}} \delta_0^\mu, \\ m^\mu = \frac{1}{\sqrt{2}\bar{r}} \left(\delta_2^\mu + \frac{i}{\sin \theta} \delta_3^\mu \right), \\ \bar{m}^\mu = \frac{1}{\sqrt{2}r} \left(\delta_2^\mu - \frac{i}{\sin \theta} \delta_3^\mu \right). \end{cases} \quad (14)$$

where $\beta = \frac{1-2\delta+4\delta^2}{\delta-1}$. A new metric is obtained by making a complex coordinates transformation

$$x^\mu \rightarrow \tilde{x}^\mu = x^\mu + iy^\mu(x^\sigma), \quad (15)$$

where $y^\mu(x^\sigma)$ are analytic functions of the real coordinates x^σ , and simultaneously let the null tetrad vectors $Z_a^\mu = (l^\mu, n^\mu, m^\mu, \bar{m}^\mu)$, with $a = 1, 2, 3, 4$, undergo the transformation

$$Z_a^\mu \rightarrow \tilde{Z}_a^\mu(\tilde{x}^\sigma, \bar{\tilde{x}}^\sigma) = Z_a^\rho \frac{\partial \tilde{x}^\mu}{\partial x^\rho}. \quad (16)$$

Obviously, one has to recover the old tetrads and metric as soon as $\tilde{x}^\sigma = \bar{\tilde{x}}^\sigma$. In summary, the effect of the "tilde transformation" (15) is to generate a new metric whose components are (real) functions of complex variables, that is

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} : \tilde{\mathbf{x}} \times \tilde{\mathbf{x}} \mapsto \mathbb{R} \quad (17)$$

with

$$\tilde{Z}_a^\mu(\tilde{x}^\sigma, \bar{\tilde{x}}^\sigma)|_{\mathbf{x}=\tilde{\mathbf{x}}} = Z_a^\mu(x^\sigma). \quad (18)$$

For our aims, we can make the choice

$$\tilde{x}^\mu = x^\mu + ia(\delta_1^\mu - \delta_0^\mu) \cos \theta \rightarrow \begin{cases} \tilde{u} = u + ia \cos \theta \\ \tilde{r} = r - ia \cos \theta \\ \tilde{\theta} = \theta \\ \tilde{\phi} = \phi \end{cases} \quad (19)$$

where a is constant and the tetrad null vectors (14), if we choose $\tilde{r} = \bar{\tilde{r}}$, become

$$\begin{cases} \tilde{l}^\mu = \delta_1^\mu \\ \tilde{n}^\mu = -\frac{1}{2} \Psi \left\{ 1 + C[(\Sigma^{-2}) \Re(\tilde{r})]^\beta \right\} \delta_1^\mu + \sqrt{\Psi} \Sigma^{\frac{\delta(2\delta+1)}{\delta-1}} \delta_0^\mu, \\ \tilde{m}^\mu = \frac{1}{\sqrt{2}(\tilde{r} - ia \cos \theta)} \left[ia(\delta_0^\mu - \delta_1^\mu) \sin \theta + \delta_2^\mu + \frac{i}{\sin \theta} \delta_3^\mu \right], \\ \tilde{\bar{m}}^\mu = \frac{1}{\sqrt{2}(\tilde{r} + ia \cos \theta)} \left[-ia(\delta_0^\mu - \delta_1^\mu) \sin \theta + \delta_2^\mu - \frac{i}{\sin \theta} \delta_3^\mu \right], \end{cases} \quad (20)$$

where $\Sigma = \sqrt{\tilde{r}^2 + a^2 \cos^2 \theta}$.

From the transformed null tetrad vectors, a new metric is recovered using (9). For the null tetrad vectors given by (20) and the transformation given by (19), the new metric, with coordinates $\tilde{x}_\mu = (\tilde{u}, \tilde{r}, \theta, \varphi)$, is

$$\begin{aligned} ds^2 = & \Sigma^{\frac{2\delta(2\delta+1)}{1-\delta}} \left[1 + C \left(\frac{\Re(\tilde{r})}{\Sigma^2} \right)^\beta \right] d\tilde{u}^2 + \\ & + 2 \frac{\Sigma^{\frac{\delta(2\delta+1)}{1-\delta}}}{\sqrt{\Psi}} d\tilde{u} \tilde{r} + 2a \Sigma^{\frac{2\delta(2\delta+1)}{1-\delta}} \times \\ & \times \left[\sqrt{\Psi} \Sigma^{\frac{\delta(2\delta+1)}{\delta-1}} - 1 - C \left(\frac{\Re(\tilde{r})}{\Sigma^2} \right)^\beta \right] \times \\ & \times \sin^2 \theta d\tilde{u} \tilde{\varphi} - 2a \frac{\Sigma^{\frac{\delta(1+2\delta)}{1-\delta}} \sin^2 \theta}{\sqrt{\Psi}} d\tilde{r} \tilde{\varphi} - \Sigma^2 d\tilde{\theta}^2 - \\ & - \left\{ \Sigma^2 + a^2 \sin^2 \theta \Sigma^{\frac{2\delta(2\delta+1)}{1-\delta}} \left[\sqrt{\Psi} \Sigma^{\frac{\delta(2\delta+1)}{\delta-1}} - \right. \right. \\ & \left. \left. + 1 - C \left(\frac{\Re(\tilde{r})}{\Sigma^2} \right)^\beta \right] \right\} \sin^2 \theta d\tilde{\varphi}^2 \end{aligned} \quad (21)$$

The form of this metric gives the general result of the Newman-Janis algorithm starting from any spherically symmetric "seed" metric. The metric given in Eq. (21) can be simplified by a further gauge transformation so that the only off-diagonal component is $g_{\varphi t}$. This procedure makes it easier to compare with the standard Boyer-Lindquist form of the Kerr metric [17] and to interpret

³ It is worth noticing that a certain arbitrariness is present in the complexification process of the functions. Obviously, we have to obtain the metric (8) as soon as $r = \bar{r}$.

physical properties such as the frame dragging. The coordinates \tilde{u} and φ can be redefined in such a way that the metric in the new coordinate system has the properties described above. More explicitly, if we define the coordinates in the following way

$$d\tilde{u} = dt + g(\tilde{r})d\tilde{r} \quad \text{and} \quad d\varphi = d\varphi + h(\tilde{r})d\tilde{r} \quad (22)$$

where

$$\begin{cases} g(\tilde{r}) = -\frac{\sqrt{\Psi}\Sigma^{\frac{2\delta^2+3\delta-2}{\delta-1}} + a^2 \sin^2 \theta}{\Psi\Sigma^2 \left[1 + C\left(\frac{\Re\{\tilde{r}\}}{\Sigma^2}\right)^\beta\right] + a^2 \sin^2 \theta} \\ h(\tilde{r}) = -\frac{a}{\Psi\Sigma^2 \left[1 + C\left(\frac{\Re\{\tilde{r}\}}{\Sigma^2}\right)^\beta\right] + a^2 \sin^2 \theta} \end{cases} \quad (23)$$

after some algebraic manipulations, one finds that, in $(t, \tilde{r}, \theta, \varphi)$ coordinates system, the metric (21) becomes

$$\begin{aligned} ds^2 = & \Sigma^{\frac{2\delta(2\delta+1)}{1-\delta}} \left[1 + C \left(\frac{\Re(\tilde{r})}{\Sigma^2} \right)^\beta \right] dt^2 + \\ & + 2a\Sigma^{\frac{2\delta(2\delta+1)}{1-\delta}} \left[\sqrt{\Psi}\Sigma^{\frac{\delta(2\delta+1)}{\delta-1}} - 1 - C \left(\frac{\Re(\tilde{r})}{\Sigma^2} \right)^\beta \right] \times \\ & \times \sin^2 \theta dtd\varphi - \left(\Psi \left[1 + C \left(\frac{\Re\{\tilde{r}\}}{\Sigma^2} \right)^\beta \right] + \right. \\ & \left. + a^2 \sin^2 \theta \right)^{-1} dr^2 - \Sigma^2 d\theta^2 - \left\{ \Sigma^2 + \right. \\ & \left. + a^2 \Sigma^{\frac{2\delta(2\delta+1)}{1-\delta}} \sin^2 \theta \left[\sqrt{\Psi}\Sigma^{\frac{\delta(2\delta+1)}{\delta-1}} - \right. \right. \\ & \left. \left. + 1 - C \left(\frac{\Re(\tilde{r})}{\Sigma^2} \right)^\beta \right] \right\} \sin^2 \theta d\varphi^2 \end{aligned} \quad (24)$$

This metric is the product of the Newman-Janis algorithm applied to the Clifton-Barrow static spherically symmetric solution of $R^{1+\delta}$ gravity, in Boyer-Lindquist type coordinates. In the same manner, axisymmetric solutions of other $f(R)$ gravity theories can be generated. This metric describes a 2-parameter family of solutions with the parametr C corresponding to the mass of the central object and a to the angular momentum per unit mass. Our result generalizes those previously obtained in Ref. [7] since, in the special case $\delta = 1/4$, not only one obtains immediately the spherically symmetric space-time

$$ds^2 = (C + r) dt^2 - \frac{1}{2} \frac{r}{C + r} dr^2 - r^2 d\Omega^2 \quad (25)$$

but, in addition, the axial metric (24) reduces to the spherically symmetric metric found in [7].

IV. CONCLUSIONS

The Newman-Janis method was developed for General Relativity but, as shown in Sec.III, it can be applied successfully also to metric $f(R)$ -gravity. The Clifton-Barrow static and spherically symmetric solution is mapped into the new stationary axisymmetric solution (24), which is the main result of this paper. Due to the fourth order of its field equations, metric $f(R)$ -gravity has a richer variety of solutions than General Relativity and the Kerr solution is not the only vacuum solution of this theory, nor the most general. The new metric testifies of this fact. In addition, it does not belong to the class of axisymmetric solutions discovered in [7].

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